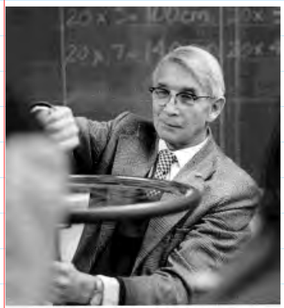
O(N) models.



Ernst Ising(1900-1998)

Ising model:

Can be defined on any finite graph $G=(V,E)$.

Configuration: $\sigma: V \to \{-1,1\}$ spins. $\beta>0$ - parameter

Probability of configuration:

$$P_\beta(\sigma) = \frac{1}{Z_\beta} e^{\beta \sum_{x\sim y} \sigma(x)\sigma(y)}, \text{ where}$$

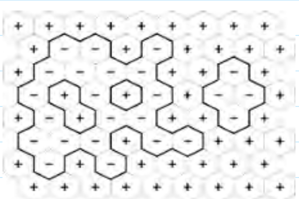
$$Z_\beta = \sum e^{\beta \sum_{x\sim y} \sigma(x)\sigma(y)} \text{ - normalization.}$$

Partition function

Equivalently:

$$P_\beta(\sigma) = \frac{1}{Z_\beta} e^{-2\beta \sum_{x\sim y} \mathbb{1}\{\sigma(x)\neq\sigma(y)\}} -$$

β - "inverse temperature"



Loop representation:

γ - collection of lattice loops.

(= boundaries of clusters with the same spin) and crosscuts

$$P_x(\gamma) = \frac{1}{Z_x} x^{\text{length of curves}}, \text{ where } x = e^{-2\beta}$$

Harry Eugene Stanley

$O(N)$ - model (loop version)

$$P_{N,x}(\gamma) = \frac{1}{Z_{N,x}} N^{\#\text{loops}} \, x^{\text{length of curves}}$$

Has to do with $(N-1)$ dimensional spins.

$N=1$, $x=1$ - Percolation

$N=1$, $x>1$ - Ising

$N=0$ - SAW (no loops).

Wouter Kager

Bernard Nienhuis

Conjecture (Kager-Nienhuis)

$\exists$ conformally invariant scaling limit for $0 \le N \le 2$

$$x = x_c(N) = \frac{1}{\sqrt{2+\sqrt{2-N}}}$$

and for $x > x_c(N)$

SLE_κ:

In the first case, $\kappa = \dfrac{4\pi}{2\pi - \arccos\left(-\frac{N}{2}\right)}$ (≤ 4)

dilute regime

In the second case, $\kappa = \dfrac{4\pi}{\arccos\left(-\frac{N}{2}\right)}$ (≥ 4)

dense regime

Let us look at dilute regime $(x = x_c)$.

Fix $a \in \partial\Omega$. Consider configurations:

loops + path from a to some $z \in \Omega$

As before, a, z are edges. = medial lattice vertices.

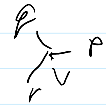
Parafermionic observable:

$$F_{N,x}(z) = \sum_{\substack{\text{all configs}\\ \text{with } \gamma \text{ joining}\\ a \text{ to } z}} N^{\#\text{loops}}\, x^{\#\text{edges}}\, e^{-i\sigma \overbrace{W_{a\to z}}^{\text{winding}}}$$

where $\sigma = \frac{1}{4} + \frac{3\kappa}{2}$, $\kappa = \frac{1}{2\pi}\arccos\frac{N}{2}$.

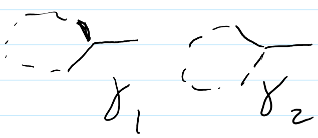
Key lemma for $O(N)$ model:

If $x = x_c$, then for any vertex v with neighboring medial vertices p, q, r, we have
$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0.$$



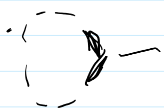
Proof. Same as the key Lemma for SAW with one difference.

SAW, case 1:



$N>0$, case 1:

γ_1, γ_2 and γ_3:



γ_3 does not exist when $N=0$.

$$c(\gamma_1) + c(\gamma_2) + c(\gamma_3) =$$

$$(q-v)\, e^{-i\sigma W_{\gamma_1}(a,q)}\, x_c^{-\ell(\gamma)} + (r-v)\, e^{-i\sigma W_{\gamma_2}(a,r)}\, x_c^{-\ell(\gamma)} + (p-v)\, N e^{-i\sigma W_{\gamma_3}(a,p)}\, x_c^{-\ell(\gamma)} =$$

$$x_c^{-\ell(\gamma)}\,(p-v)\, e^{-i\sigma W_{\gamma_3}(a,p)} \Big(\underbrace{e^{i\frac{2\pi}{3}}}_{=\left(\frac{q-v}{p-v}\right)} e^{-i\sigma(-\frac{4\pi}{3})} + \underbrace{e^{-i\frac{2\pi}{3}}}_{=\left(\frac{r-v}{p-v}\right)} e^{-i\sigma(\frac{4\pi}{3})} + N \Big) = 0,$$

by our choice of σ.

Case 2 is the same as for SAW ∎

As in SAW, the integral in dual lattice is 0, not enough to determine F!

Conjecture. Let Ω be a simply connected domain, $a \in \partial\Omega$, $\varphi : \Omega \to H$ – unique conformal map with $\varphi(z) = \frac{e^{i\theta}}{z-a} + ib + g(z)$ near a.

Let us consider a lattice approximation Ω_δ of Ω (by a fairly general lattice). Then

$$\delta^{-\sigma} F_{\delta, N, x_c}(z) \to (\varphi'(z))^{\sigma}.$$

Theorem (Smirnov) For $N=1$, $x_c = \frac{1}{\sqrt{3}}$ (critical Ising) the conjecture holds.